Master‘s Program in Medical Physics

Biomedical Engineering

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What is Biomedical Engineering?

system, apparatus, instruments, programs

for

detection, therapy, supervision, prevention of diseases

and

compensation and facilitate disabilities.

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Examples

- ECG: measuring the electrical heart activities
- blood pressure measurement
- oxygenation of blood
- wheelchair
- operation theatre
- ...

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10.09.07	Heiler	Repeat Basic Knowledge
17.09.07	Heiler	Blood Pressure
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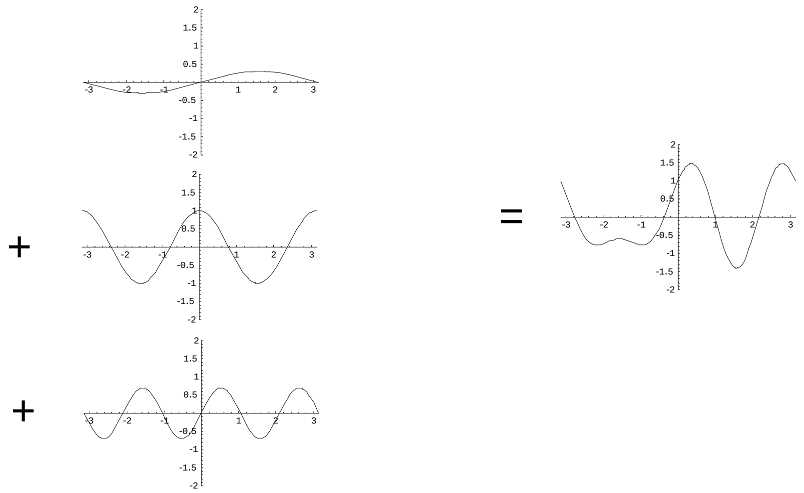
01.10.07	Heiler	Blood Flow
08.10.07	Zöllner	ECG
15.10.07	Zöllner	EEG
22.10.07	Zöllner	MEG
29.10.07	Zöllner	MEG
05.11.07	Kirsch	MRS
12.11.07	Kirsch	MRS
19.11.07	Kirsch	MRS
26.11.07	Kirsch	MRS

Repetition of Basic Knowledge

- Fourier analysis
- electric networks

Fourier Transform



Fourier Transform

The Fourier Transform of f(x) is defined as

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xs} dx.$$

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-ixs} dx.$$

This integral, which is a function of s may be written F(s). Transforming F(s) by the same formula, we have*

$$f(x) = \int_{-\infty}^{\infty} F(s) e^{i2\pi xs} ds.$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{ixs} ds.$$

If f(x) and F(s) are a transform pair in system 1, then f(x) and F(s/2π) are a Transform pair in system 2…

*… The second transformation is not exactly the same as the first. In this form, two succesive transformations yield the original funcion.

Box Function

Example: Box Function

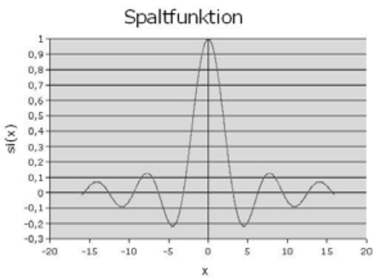
$$\cos kx = \frac{e^{jkx} + e^{-jkx}}{2}$$

$$\sin kx = \frac{e^{jkx} - e^{-jkx}}{2j}$$

$$f(t) = \begin{cases} 1 & -1 < x < 1 \\ 0 & otherwise \end{cases} \qquad F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

$$F(\omega) = \int_{-1}^{1} e^{i\omega t} dt = \frac{1}{i\omega} e^{i\omega t}\Big|_{-1}^{1} = \frac{1}{i\omega}(e^{i\omega} - e^{-i\omega}) = \frac{1}{i\omega}(2i \sin\omega)$$

$$F(\omega) = 2\frac{\sin\omega}{\omega} = 2 \cdot \mathrm{sinc}(\omega)$$



Further Examples

$$f(x) = e^{-x^2/2} \Longleftrightarrow f(\nu) = \sqrt{2\pi}e^{-\nu^2/2}$$

$$\mathrm{f(x)} = \begin{cases} \text{"}\infty\text{"} & \mathrm{x} = 0 \\ 0 & \text{otherwise} \end{cases} \Longleftrightarrow f(w) = 1$$

Properties

The shift property:

$f(x) \leftrightarrow F(k)$	Function and Fourier transform.
$f(x-u) \leftrightarrow e^{iku}F(k)$	Delay leads to phase shift in Fourier space.

$w = x - u$	$dx = dw$	
$\int_{-\infty}^{\infty} f(x-u)e^{ikx}dx = \int_{-\infty}^{\infty} f(w)e^{ik(w+u)}dw = e^{iku}F(k)$		Proof by substitution.

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Properties

Linearity: $FT[a\ f(x) + b\ g(x)] = a\ FT[f(x)] + b\ FT[g(x)]$

$$\int_{-\infty}^{\infty} [af(x) + bg(x)] e^{-ikx} dx = a\int_{-\infty}^{\infty} f(x) e^{-ikx} dx + b\int_{-\infty}^{\infty} g(x) e^{-ikx} dx = aFT[f(x)] + bFT[g(x)]$$

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Convolution

The convolution of two functions f(x) and g(x) is: $\int_{-\infty}^{\infty} f(\tau) \cdot g(x-\tau) d\tau = f(x) \otimes g(x)$

$$FT[f(x) \otimes g(x)] = FT[f(x)] \cdot FT[g(x)]$$

Proof: $FT[f(x) \otimes g(x)] = FT\left[\int f(x-\tau) g(\tau) d\tau\right]$

$$= \iint f(x-\tau) g(\tau) d\tau \cdot e^{-ikx} dx$$

substitution: $x' = x - \tau$, $dx' = dx$

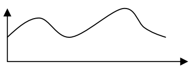
$$= \iint f(x') g(\tau) d\tau\ e^{-ik(x'+\tau)} dx'$$

$$= \int f(x') e^{-ikx'} dx' \int g(\tau) e^{-ik\tau} d\tau$$

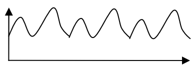
$$= FT[f(x)] \cdot FT[g(x)]$$

Properties

•continuous functions

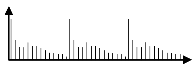


•periodic functions

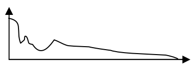


•discrete functions

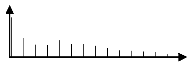
•periodic, discrete functions



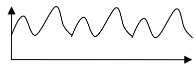
•continuous functions



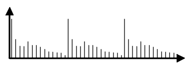
•discrete functions



•periodic functions



•periodic, discrete functions



Fourier Transform of Differential Equation

$$\frac{\partial}{\partial t} f(t) = \frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \right) = \int_{-\infty}^{\infty} F(\omega) \frac{\partial}{\partial t} e^{i\omega t} d\omega = \int_{-\infty}^{\infty} i\omega F(\omega) e^{i\omega t} d\omega$$

- Consequence:
  - f‘: amplitude is multiplied by iω
  - a differential equation becomes an algebraic equation in the Fourier space

Fourier Transform of Integral Equation

$$\int f(t)dt = \int \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega dt = \int_{-\infty}^{\infty} F(\omega)\int e^{i\omega t}dt\, d\omega = \int_{-\infty}^{\infty} \frac{1}{i\omega} F(\omega)e^{i\omega t}d\omega$$

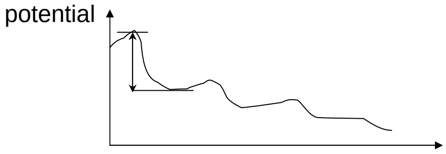
- Conclusion:
  - Integral: divide amplitude by iω
  - an integral equation becomes an algebraic equation in the Fourier space.

Electric Networks

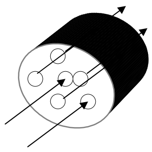
Electric Networks

Electric Measurement Variables

- voltage U
  - measured between two points
  - each point has a potential (like height of mountain)
  - difference between both potentials = voltage

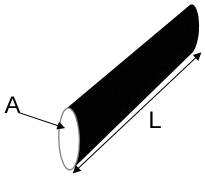


- current I
  - charge transport along a wire
  - = charge/time



- Ohm resistance

$$R = \frac{\rho L}{A}$$



ρ: specific resistance,
L: length,
A: cross section

Capacitor

- electric field $E = -\frac{\partial \phi}{\partial x}$
- force $F = qE$

- plate capacitor

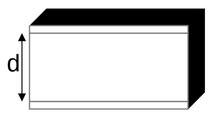
$C = \varepsilon_0 \varepsilon_{dielectric} \frac{A}{d}$

ε_0 : constant

$\varepsilon_{dielectric}$: dielectricity of material between plates

A: cross section

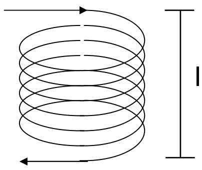
d: distance between plates



$$E = \frac{U}{d}$$

$$C = \frac{Q}{U}$$

Coil



radius r

cross section A

$$B = \mu_0 \cdot n \cdot I$$

$$\Phi_m = BA = \mu_0 \cdot n \cdot I \cdot A$$

$$\frac{d\Phi_m}{dt} = \mu_0 \cdot n \cdot A \cdot \frac{dI}{dt}$$

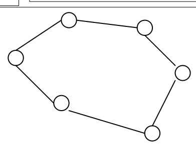
$$U_{ind} = -N \cdot \frac{d\Phi_m}{dt} = \mu_0 \cdot n^2 \cdot l \cdot A \cdot \frac{dI}{dt} = -L \frac{dI}{dt}$$

➔ $L = \mu_0 \cdot n^2 \cdot V$

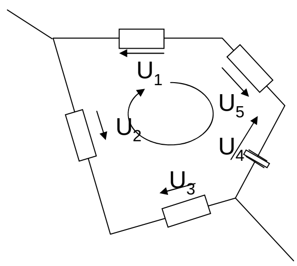
N: windings, N=nl
μ: permeability (material constant)

Kirchhoff Rules

- mesh rule

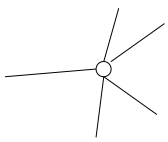


$-U_1-U_2+U_3-U_4+U_5=0$

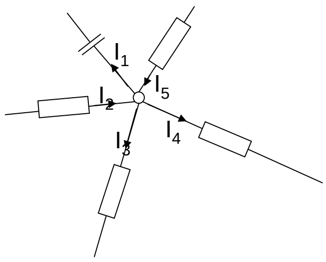


  - the sum of all voltages in a closed loop is 0
    - Imagine walking around a mountain. Round trip ends at the same height, i.e. sum of heights over the way = 0.

- node rule



$-I_1+I_2-I_3-I_4+I_5=0$



  - the sum of all currents in a node is 0
    - the sum of charged particles coming in = sum of particles leaving

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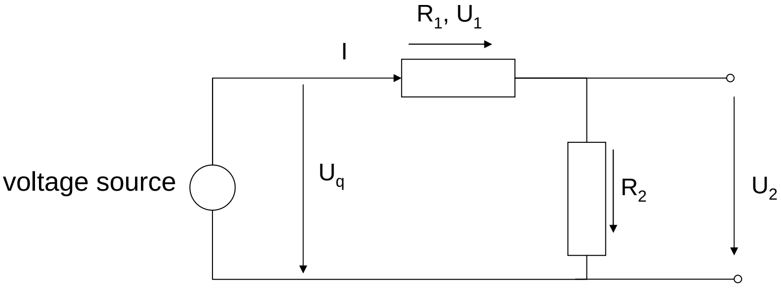
Voltage Divider



mesh rule: $U_q = U_1 + U_2$

$$\frac{U_2}{U_q} = \frac{U_2}{U_1 + U_2} = \frac{U_2 / I}{U_1 / I + U_2 / I} = \frac{R_2}{R_1 + R_2}$$

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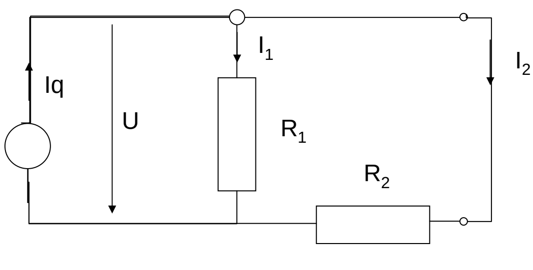
Current Divider



node rule: $I_q = I_1 + I_2$

$$U_1 = U_2 = U$$

$$\frac{I_2}{I_q} = \frac{I_2}{I_1 + I_2} = \frac{I_2 / U}{I_1 / U + I_2 / U} = \frac{1/R_2}{1/R_1 + 1/R_2}$$

Alternating current

An AC voltage

$$U(t) = U_0 \cdot \cos \omega t$$

which is applied to a resistance R causes a AC current

$$I(t) = I_0 \cdot \cos \omega t \ .$$

Power:

$$P_{el} = U \cdot I = U_0 I_0 \cos^2 \omega t$$

$$\overline{P}_{el} = \frac{1}{T}\int_0^T U_0 I_0 \cos^2 \omega t \, dt = \frac{1}{2} U_0 I_0 \quad \Rightarrow \quad U_{eff} = \frac{U_0}{\sqrt{2}}, \quad I_{eff} = \frac{I_0}{\sqrt{2}}$$

Inductivity

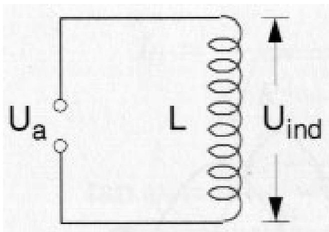
$$U_a + U_{ind} = 0$$

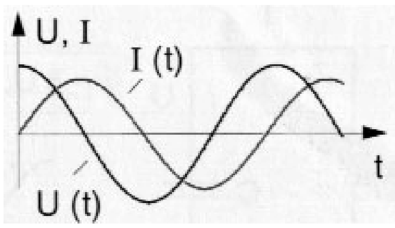
$$U_0 \cdot \cos \omega t = L \frac{dI}{dt},$$

$$\Rightarrow \quad I = \frac{U_0}{L}\int \cos \omega t \, dt = \frac{U_0}{\omega L} \sin \omega t$$

$$= I_0 \cdot \sin \omega t$$

$$|R_L| = \frac{U_0}{I_0} = \omega \cdot L$$





Capacity

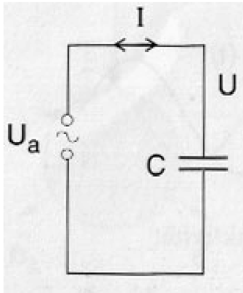
$$U = \frac{Q}{C}$$

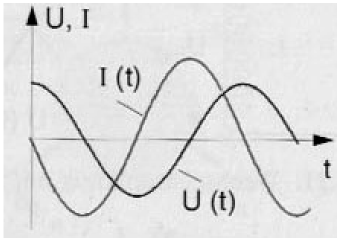
$$\frac{dU}{dt} = \frac{1}{C}\frac{dQ}{dt} = \frac{1}{C} \cdot I.$$

➔ $$I = -\omega C \cdot U_0 \cdot \sin \omega t$$

$$= -I_0 \cdot \sin \omega t$$

$$|R_C| = \frac{U_0}{I_0} = \frac{1}{\omega C}$$





Impedance

- Z=U/I is a complex value if L or C is in network
- assume

example: capacity

$$U = \frac{Q}{C} = \frac{1}{C}\int I\,dt$$

$$\Rightarrow \tilde{U} = \frac{1}{C}\frac{\tilde{I}}{i\omega}$$

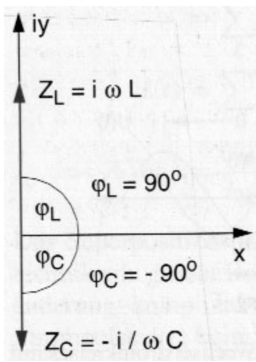
$$Z = \frac{U}{I} = \frac{\tilde{U}}{\tilde{I}} = \frac{1}{i\omega C}$$

example: inductivity

$$U = L\frac{dI}{dt}$$

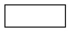
$$\tilde{U} = L \cdot i\omega\tilde{I}$$

$$Z = \frac{U}{I} = \frac{\tilde{U}}{\tilde{I}} = i\omega L$$



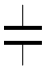
Ohm/Capacitive/Inductive Resistance

- Ohm resistance
  - the Ohm resistance is constant over time and defined by
    Z = U/ I



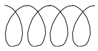
- capacitive resistance
  - from the definition we have the capacitive resistance

$$Z = \frac{1}{i\omega C}$$



- inductive resistance
  - from the definition we have the inductive resistance

$$Z = i\omega L$$



Resistance Networks

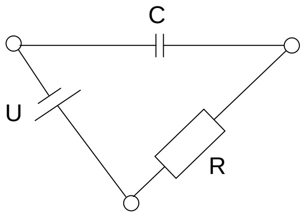
- description of differential equations
  - example: sequence of Ohm, capacitive, and inductive resistor in series

$$U = RI + \frac{1}{i\omega C} I + i\omega L \cdot I$$

  - example: Ohm, capacitive, and inductive resistor in parallel

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{i\omega L} + i\omega C$$

Example



mesh rule:

$$U = \frac{1}{j\omega C} I + RI$$

$$Z = \frac{1}{j\omega C} + R$$

Important

- definition of Fourier Transform and its main properties:
  - linearity, convolution theorem, differential and integral equations in the Fourier space
- electrical measurement values
  - voltage, current, power, different impedance: Ohm, Capacitor, Inductivity (+their symbols), symbols for current or voltage sources
  - Kirchhoff rules (node and mesh rule) and being able to apply them to different electronic circuits

Overview

- Fluid Parameters: Pressure, Flow
- Fluids in Motion
- Flow of Fluids in Tubes
- Blood Pressure
- Measurement of Blood Pressure
- Pressure Sensor

Fluid Parameters

parameter	**Formula/Symbol**	**SI unit**
Density:	$\rho = m/V$	[kg/m^3]
Temperature	T	[K]
Velocity	$\mathbf{v} = \frac{d\,\mathbf{s}}{dt} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$	[m/s]
Pressure	$p = F/A$	[N/m^2]

Pressure Conversion Factors

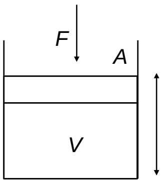
	Atmosphere	N/m^2 = Pa	mm Hg	mm H_2O
Atmosphere	1	1.01×10^5	760	10300
N/m^2 = Pa	9.87×10^{-6}	1	0.0075	0.102
mm Hg	0.00132	133	1	13.6
mm H_2O	9.86×10^{-5}	9.81	0.0735	1

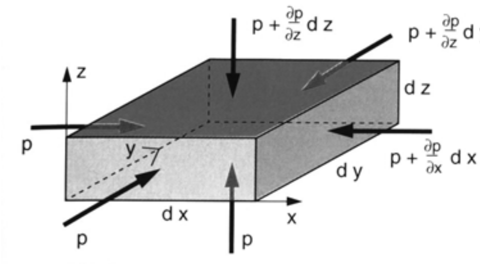
Pressure in the Body

Site		Pressure (mm Hg)
Arterial blood pressure:	systole	100–140
	diastole	60–90
Capillary blood pressure:	arterial end	~30
	venous end	~10
Venous blood pressure:	smaller veins	3–7
	great veins	<1
Cerebrospinal pressure in brain (lying down)		5–12
Gastrointestinal pressure		10–20
Bladder pressure		5–30
Lungs:	during inspiration	minus 2–3
	during expiration	2–3
Intrathoracic cavity (between lung and chest wall)		minus 10
Joints in skeleton		up to 10 000
Foot pressure:	static	up to 1200
	dynamic	up to 7500
Eye		12–23
Middle ear		<1

Static Pressure

$p_{stat} = F/A;$





$$F_x = p \cdot dydz - \left(p + \frac{\partial p}{\partial x} dx \right) dydz = -\frac{\partial p}{\partial x} dV \qquad F_y = -\frac{\partial p}{\partial y} dV \quad , \quad F_z = -\frac{\partial p}{\partial z} dV$$

$$\vec{F} = -grad\ p \cdot dV$$

Hydrostatic Pressure

Force of Gravity of volume element dV: $dF = \rho \cdot g \cdot dV, \quad \Rightarrow \quad dp = \rho \cdot g \cdot dz$

pressure on ground element da: $p_{atm}(0) = \int_0^H \rho \cdot g \cdot dz = \rho \cdot g \cdot H$

➔ hydrostatic pressure distribution: $p_{atm}(h) = \int_h^H \rho \cdot g \cdot dz = \rho \cdot g (H - h)$

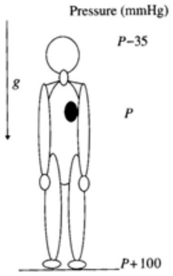
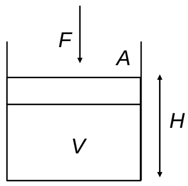


Figure 2.1. *Hydrostatic pressure when standing.*

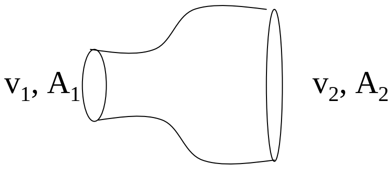


Flow

flow: $Q = dV/dt = Av$

⇧

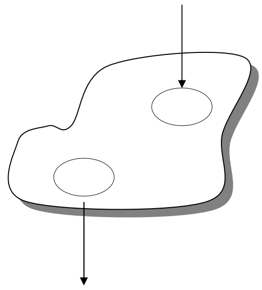
$dV = A\,ds = A\,v\,dt$



incompressible flow

continuity law: $Q = v_1 A_1 = v_2 A_2 = const$

general case: $Q = V_1 A_1 + Q_{source} - Q_{drain}$



Bernoulli-Equation

Potential Energy: $\Delta W_1 = F_1 \Delta x_1 = p_1 A_1 \Delta x_1 = p_1 V_1$

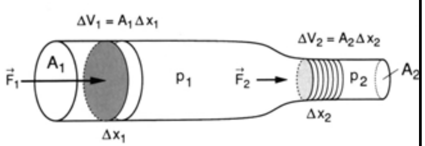
$\Delta W_2 = p_2 V_2$

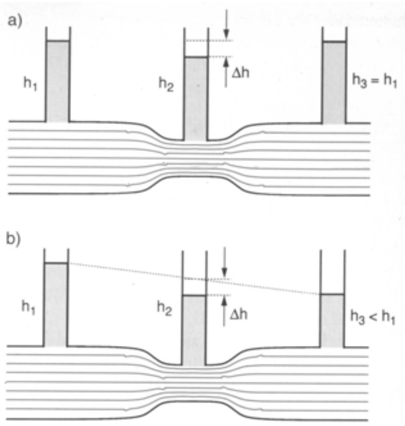
Kinetic Energy: $E_{kin} = \frac{1}{2} m v^2 = \frac{1}{2} \rho v^2 V$

Conservation of Energy:

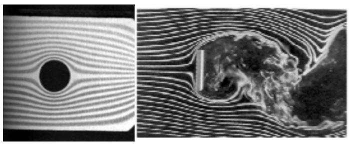
$$p_1 V_1 + \frac{1}{2}\rho v^2 V_1 = p_2 V_2 + \frac{1}{2}\rho v^2 V_2$$

$$p_1 + \frac{1}{2}\rho v^2 = p_2 + \frac{1}{2}\rho v^2 = const.$$



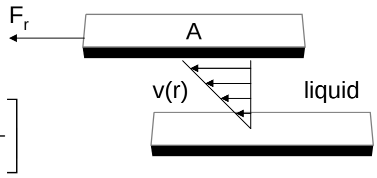


Fluids in Motion I



$$F_F = \eta \cdot A \frac{dv}{dr}$$

η ... viscosity $\left[\frac{NS}{m^2}\right]$

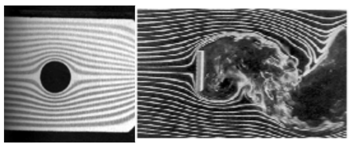


Examples:

fluid	η [mNs/m²]=[mPas]
water	1.002
blood	approx. 3-5
glycerin	1480
mercury	1.55

T = 20°

Fluids in Motion II



laminar current: frictional force > accelerating Force

otherwise formation of turbulences possible (where velocity gradient is strong)

distinction: Reynolds number: $\mathrm{Re} = \frac{\rho \cdot v \cdot D}{\eta}$ D ... characteristic length
... tube diameter

- laminar flow: Re < 2300
- transient: 2300 < Re < 4000
- turbulent flow: Re > 4000

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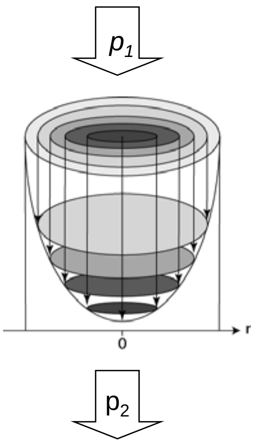
Flow of Fluids in Tubes I



Due to symmetry the velocity of the current is only depended on the distance to the axis!

Frictional force and resulting pressure force on the end surfaces are in equilibrium.

➔ $$F_F = 2\pi r L \cdot \eta \frac{dv}{dr} = \pi r^2 (p_1 - p_2) = F_p$$

⟷ $$\frac{dv}{dr} = \frac{p_1 - p_2}{2\eta L} r$$

➔ $$v(r) = \int_r^R \frac{p_1 - p_2}{2\eta L} r dr + C = \frac{p_1 - p_2}{4\eta L} \cdot (R^2 - r^2)$$

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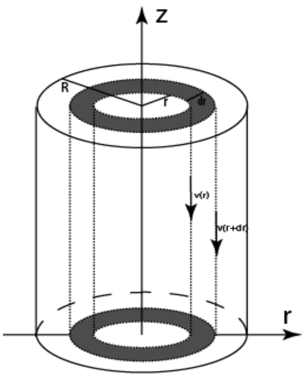
Flow of Fluids in Tubes II



Flow: $$Q = \int_0^R v(r) dA = \int_0^R v(r) 2\pi r dr$$

$$= \int_0^R \frac{2\pi r (p_1 - p_2)}{4\eta L} (R^2 - r^2) dr$$

$$= \frac{\pi R^4 (p_1 - p_2)}{4\eta L} - \frac{\pi R^4 (p_1 - p_2)}{8\eta L}$$

$$= \frac{\pi R^4 (p_1 - p_2)}{8\eta L}$$ Poiseuille´s equation

Resistance: $$K = \frac{p_2 - p_1}{LQ} = \frac{8\eta}{\pi R^4}$$